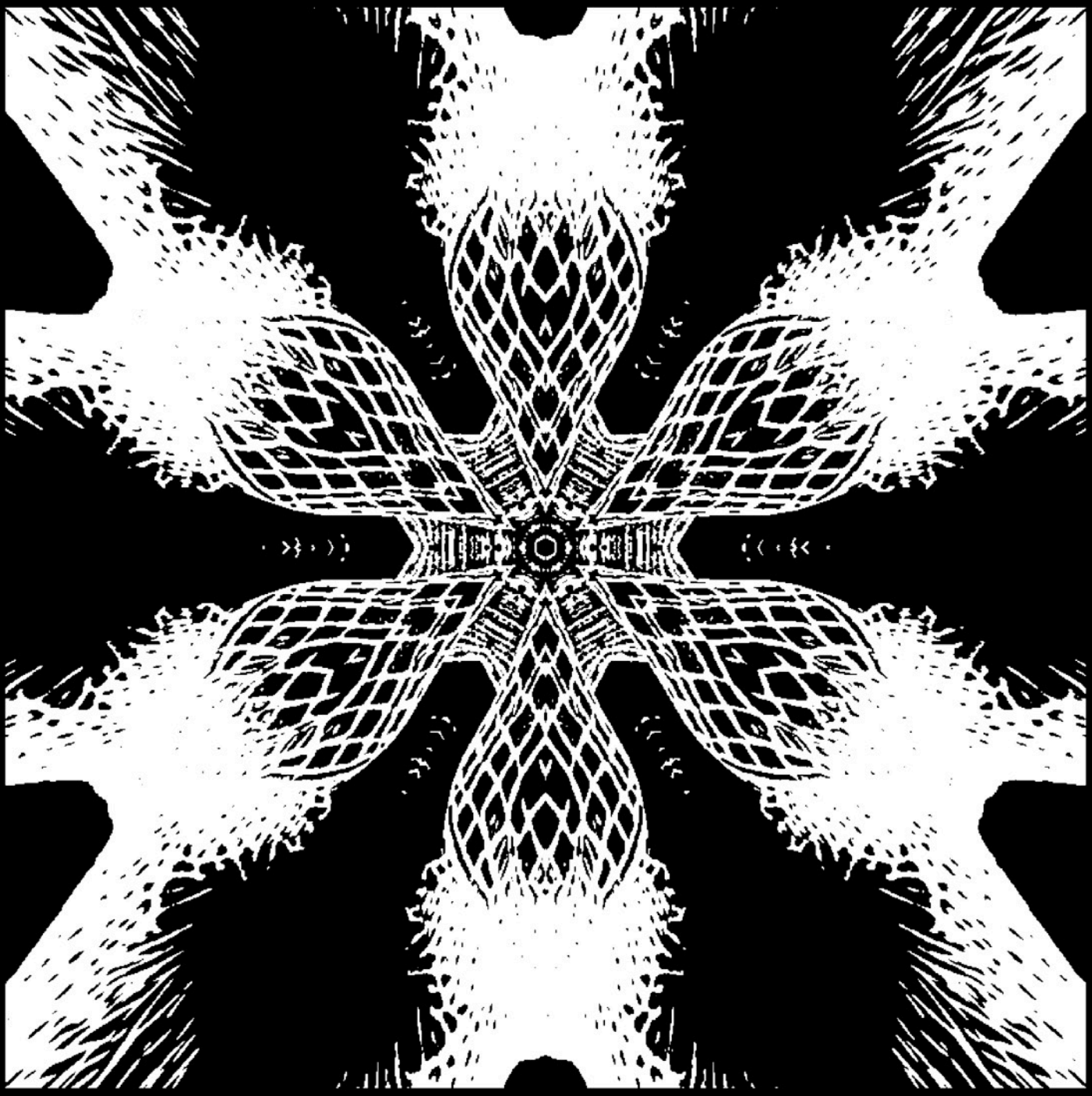

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A Constraint-Based Research Program for the Global Regularity of Navier–Stokes

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GEOMETRIC DEPLETION AND THE VISCOSITY FLOOR

A Constraint-Based Research Program for the Global Regularity of Navier–Stokes

By Sum Bunny

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Abstract

The global regularity of the 3D incompressible Navier–Stokes equations depends on whether the nonlinear vortex stretching mechanism is intrinsically depleted by geometric constraints. Building on the coherence criteria of Constantin and Fefferman [CF93], this paper outlines a research program centered on a **Directional Viscosity Floor**: a hypothesis that, in high-vorticity cores, the direction field of vorticity is stabilized by a projected drift–diffusion structure that prevents the loss of Hölder coherence required for geometric depletion.

To test this idea in a rigorous simplified setting, we introduce and analyze a "direction–diffusion" toy model that isolates the competition between supercritical scalar amplitude growth and geometric decorrelation caused by directional diffusion. The model admits a theorem demonstrating threshold elevation: sufficiently strong directional diffusion enforces a uniform alignment bound strictly below perfect strain tracking, thereby converting the amplitude dynamics from supercritical growth to subcritical decay.

Status of Results and Scope of Claims

This manuscript presents a research program and an analytical reduction, not a proof of global regularity. The status of the results is as follows:

- **Theorem 4.1 (Toy Model): Proven.** This theorem establishes that in a simplified setting with passive strain, directional diffusion is sufficient to deplete nonlinearity.
- **Theorem 3.2 (Target Continuation Criterion): Conditional.** This theorem formulates a sufficient condition for regularity based on specific integrability bounds on the logarithmic gradient of vorticity.

- **Lemma A.2 (Core Drift Bound): OPEN.** This is the central analytical bottleneck identified by the program. No proof is provided that solutions to the Navier–Stokes equations satisfy this bound.
- **Goal:** The purpose of this paper is to isolate the drift bound in Lemma A.2 as a concrete target for future analysis.

1. THE SUPERCRITICAL SCALING OBSTRUCTION

We consider the 3D incompressible Navier–Stokes equations on $\mathbb{R}^3$:

$$\partial_t u + (u \cdot \nabla)u = -\nabla p + \nu \Delta u$$

$$\nabla \cdot u = 0$$

with vorticity $\omega = \nabla \times u$.

The central analytical difficulty is the supercriticality of the natural energy control. Under the scaling $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$, the L^2 energy scales subcritically and therefore cannot prevent concentration leading to potential blow-up.

A scale-critical quantity is the enstrophy:

$$\Omega(t) = \tfrac{1}{2} \|\omega(\cdot,t)\|^2 \quad (L^2)$$

Formally differentiating yields the enstrophy balance:

$$\frac{d}{dt}\Omega(t) + \nu \|\nabla\omega(\cdot,t)\|^2 = \int (S\omega) \cdot \omega \, dx$$

where $S = \tfrac{1}{2}(\nabla u + \nabla u^T)$ is the strain tensor. The right-hand side represents vortex stretching. Without geometric cancellation, it can dominate dissipation at high frequencies.

2. GEOMETRIC DEPLETION (THE CONSTRAINT REGION)

We posit that the dynamics may be constrained to a region of phase space in which vortex stretching is depleted by geometric structure.

2.1 Functional framework

Let $\xi(x,t) := \omega(x,t) / |\omega(x,t)|$ be the unit direction field of vorticity wherever $|\omega| > 0$.

Constantin and Fefferman [CF93] established that if ξ remains locally coherent (e.g., Hölder continuous) in regions where vorticity is large, then the stretching term is depleted.

Lemma 2.1 (Geometric depletion [CF93]).

Assume that in regions of sufficiently high vorticity, the direction field satisfies a local Hölder coherence condition:

$$|\xi(x,t) - \xi(y,t)| \le C|x-y|^{a}$$

Then the stretching term admits an estimate that is subcritical relative to dissipation (see [CF93], Theorem 1, or Theorem 2 in the localized formulation):

$$\left| \int (S\omega) \cdot \omega \right| \le \varepsilon \nu \|\nabla\omega\|^2 + C_\varepsilon \|\omega\|^2 \Phi(t)$$

Consequently, sufficient directional coherence implies that vortex stretching cannot drive finite-time blow-up. This motivates the program: the primary obstruction is not controlling $|\omega|$ directly, but controlling the **geometry** encoded by ξ.

3. THE VISCOSITY FLOOR HYPOTHESIS

The open problem is justifying coherence of ξ in the regimes relevant to blow-up. We propose a mechanism driven by viscosity at the smallest scales.

Hypothesis 3.1 (Directional viscosity floor; program statement).

In high-vorticity cores, the evolution of the direction field ξ is governed by a projected drift–diffusion structure in which the effective drift is scale-subcritical and the reaction term is locally integrable. In such a regime, the direction field acquires a local Hölder modulus of continuity, activating geometric depletion.

3.1 Physical motivation: the projected direction equation

Writing the vorticity equation and enforcing the constraint $|\xi|=1$ via the orthogonal projection $P_\xi = I - \xi \otimes \xi$, one obtains the exact identity (see e.g., Constantin–Fefferman–Majda, *Comm. Pure Appl. Math.*, 1996):

$$\partial_t \xi + (b \cdot \nabla)\xi = \nu P_\xi(\Delta\xi) + P_\xi(S\xi)$$

where the effective drift velocity is:

$$b = u + 2\nu \nabla \log|\omega|$$

The additional drift component $2\nu \nabla \log|\omega|$ has critical dimensional scaling (L/T). However, we hypothesize that within coherent high-vorticity cores, the logarithmic gradient of vorticity may satisfy scale-subcritical spacetime integrability bounds.

Heuristic confinement: In canonical filament geometries (e.g., Gaussian-like vorticity magnitude profiles), $\nabla \log|\omega|$ points radially inward, suggesting that the drift may act as a confinement mechanism rather than a random destabilizer.

3.2 The Logical Roadmap and Formal Target

The analytical reduction proposed here follows a strict dependency chain.

[Logic Diagram]

(A) Core Drift & Strain Bounds (Lemma A.2 + A.1)

(Integrable drift + mild strain control)

⬇

(B) Hölder Coherence of ξ (Lemma A.1)

(Viscosity floor stabilizes direction field)

⬇

(C) Geometric Depletion (Lemma 2.1)

(Spatially coherent vorticity suppresses nonlinear stretching)

⬇

(D) Subcritical Enstrophy Inequality

(Dissipation dominates stretching)

⬇

(E) Continuation Criterion (Theorem 3.2)

Theorem 3.2 (Target Continuation Criterion — Program Formal Target).

Let u be a Leray–Hopf weak solution to the 3D Navier–Stokes equations on (0,T).

If the local logarithmic gradient of vorticity satisfies the subcritical integrability bounds of Lemma A.2 **and** the strain satisfies the mild control of Lemma A.1 on the high-vorticity set Ω_R(t),

Then the solution u(t) does not blow up at time T and can be continued to T + δ.

Reader's Guide / How to Use This Paper

For researchers interested in proving global regularity using this framework, the analysis reduces to a single focal point:

- **The Bottleneck:** The critical open problem is proving **Lemma A.2** (Appendix A.2). Establishing the integrability of **$\nabla \log|\omega|$** in high-vorticity regions is the necessary step to close the loop.
- **The Mechanism:** Appendix A.3 suggests that boundary-layer tail estimates (rigidity of the core profile) are the most promising path to establishing Lemma A.2.
- **Structure:** Everything else in this paper (Theorem 4.1, Lemma A.1) is supporting infrastructure that assumes Lemma A.2 holds.

4. THE DIRECTION–DIFFUSION MODEL (SYSTEM 4.1)

To demonstrate the viscosity floor mechanism rigorously in a simplified setting, we analyze a toy system isolating the competition between scalar growth and directional diffusion.

4.1 Notation and setting

Let $\mathbb{T}^3 = (\mathbb{R}/\mathbb{Z})^3$ be the 3D flat torus. Fix a smooth strain axis field $\mathbf{e} \in W^{2,\infty}(\mathbb{T}^3; \mathbb{S}^2)$ **(viewed as a unit direction field modulo sign)**. Let $\nu, \nu_d > 0$. Define the alignment functional $\mathbf{m(t) := \int (\xi \cdot e)^2\, dx}$ and the strain correlation spectral gap $\boldsymbol{\lambda_* := \lambda_{max}(\int e \otimes e\, dx)}$. Note that $\lambda_* < 1$ for non-constant e.

4.2 System definition

$$\mathbf{da/dt = a(t)^2\, m(t) - \nu\, a(t)}$$

$$\mathbf{\partial_t \xi = \nu_d\, \Delta \xi + a(t)\, P_\xi\, e(x)}$$

4.3 Theorem: Regularity via geometric depletion

Theorem 4.1 (Viscosity floor / threshold elevation).

Assume $\lambda_* < 1$. Fix $\eta > 0$ and $a_0 > 0$. There exists a critical diffusivity ν_d^* such that if $\nu_d \geq \nu_d^*$, the solution exists globally and satisfies $m(t) \leq \lambda_* + \eta$. If $a_0 < \nu / (\lambda_* + \eta)$, then $a(t) \to 0$.

Proof Sketch.

The proof relies on a bootstrap argument coupled with the smoothing properties of the heat semigroup. The direction equation implies that deviations from the spatial mean decay via the heat semigroup $e^{\nu_d t \Delta}$. For sufficiently large ν_d, the direction field $\xi(x,t)$ relaxes rapidly toward spatial constancy, while the fixed strain field $e(x)$ retains spatial variance. Consequently, the alignment functional $m(t)$ is dominated by the spectral gap $1 - \lambda_*$ of the strain field, strictly bounding $m(t) < 1$. ∎

4.4 Remark on Active Strain

In the full NSE, S is coupled to ω. However, since S is a singular integral of ω (S ∈ BMO vs ξ ∈ C^α), the strain field **cannot in general be expected to track** the direction field with perfect phase accuracy in the ultraviolet limit. It lacks the regularity structure required to maintain perfect alignment against a Hölder-continuous direction field.

5. DISCUSSION AND CONCLUSION

5.1 Conclusion

This work isolates a minimal geometric bottleneck: the regularization of the vorticity direction field by viscosity. We have shown that under scale-subcritical drift assumptions, this mechanism is sufficient in principle to deplete vortex stretching.

5.2 Relation to Known Regularity Criteria

This program complements existing criteria but differs in its target variable:

- **Prodi–Serrin:** Requires **u** ∈ **L^p L^q**. Our criterion is similarly scale-invariant but targets the geometry of ω.
- **Beirão da Veiga:** Requires **S** ∈ **L¹ L^∞**. Our approach permits S to be large in magnitude, provided it is misaligned with ξ.
- **Caffarelli–Kohn–Nirenberg:** Establishes partial regularity. Our program targets the internal structure of the singular set.

5.3 Minimal Structure of a Counterexample

If a finite-time singularity occurs, this framework implies that the solution must violate at least one of the structural hypotheses. A blow-up requires either:

1. **Drift Explosion:** The logarithmic gradient ∇ log|ω| fails to be integrable (e.g., vortex sheets or fractal intermittency).
2. **Strain Locking:** The strain field S aligns perfectly with ξ despite the regularity mismatch.
3. **Coherence Failure:** The direction field ξ loses Hölder continuity at the blow-up time.

5.4 What Remains Open

To complete this program and convert the conditional criterion into a regularity proof, the following tasks remain open:

- **Proof of Lemma A.2:** Establishing the L^6L^9 bound on **∇ log|ω|** from the Navier–Stokes equations.
- **Profile Rigidity:** Deriving the tail bounds in Appendix A.3 directly from dissipation estimates.
- **Strain Control:** Verifying the L^1L^∞ strain assumption in the blow-up regime.

Invitation

The author hopes this framework clarifies the geometric dependencies of the problem and serves as a useful roadmap for the community. The analytical reduction presented here is intended to be a tool for future researchers, regardless of the authorship of any eventual solution.

APPENDIX A: Conditional Coherence

A.1 Conditional Coherence Lemma

Lemma A.1. If $\mathbf{b} \in L^p L^q$ (subcritical) and $\mathbf{S} \in L^1 L^\infty$, then $\xi \in C^\alpha_{loc}$.

Proof. Follows from De Giorgi–Nash–Moser theory for parabolic systems with subcritical drift.

A.2 Core Drift Bound: A Concrete Analytic Target

Define $G_R = \chi_R(|\omega|) \nabla \log|\omega|$.

Lemma A.2. If $G_R \in L^6(0,T; L^9(\Omega_R(t)))$, then the drift is subcritical.

Technical Note: The definition of G_R restricts estimates to $|\omega| \geq R/2$, excluding the nodal set $\{\omega=0\}$. The analysis applies to each connected component of Ω_R independently and does not require global topological assumptions.

A.3 Boundary-Layer Rigidity

Lemma A.3. If G_R satisfies a tail bound with exponent $\beta > 9$, then $G_R \in L^6 L^9$.

Remark: The exponent $\beta > 9$ is chosen for concreteness relative to the dimension $d=3$. The result holds for any $\beta > d$ with appropriate adjustment of the constants, indicating that any polynomial decay of the gradient distribution stronger than dimensional scaling is sufficient.

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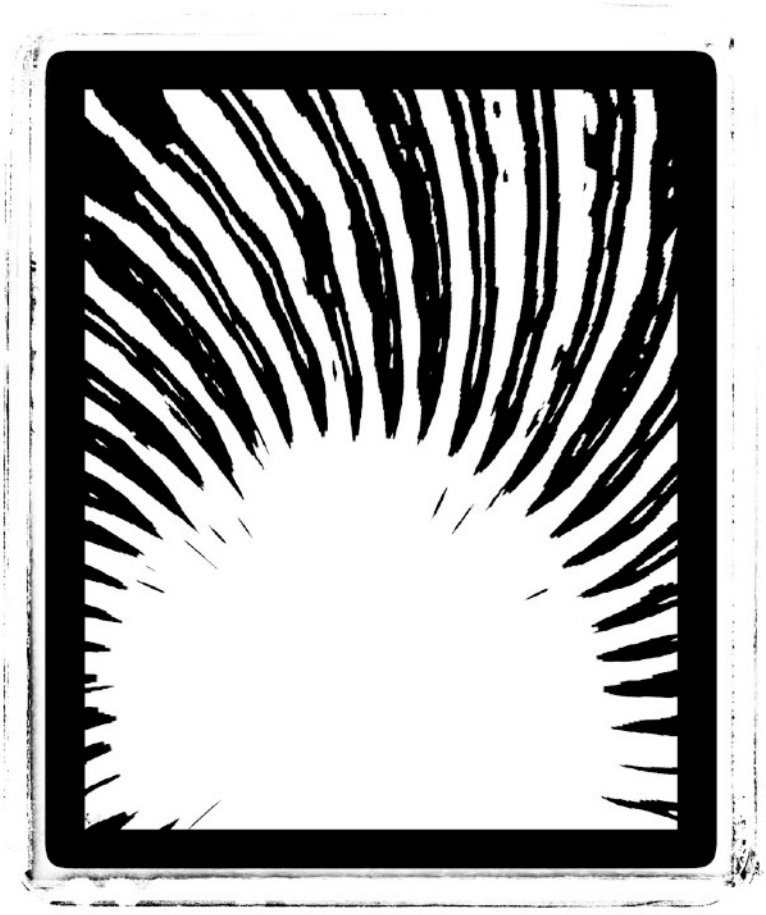
Part of the Sum Bunny Canon (Ark Hive Press).

This was written through a collaboration between a human author and AI systems acting as analytical tools. The machines provided structure; the human provided judgment. Nothing appears here that the author did not personally create, refine, choose, and claim.

Hey Josiane,
Soon you will rise and shine!
Sum Bunny Loves You!!!



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